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Determination of Subdominant Solutions Using a Partial Wronskian

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ABSTRACT

A simple step-by-step technique is presented for the numerical integration of certain difficult-to-obtain subdominant solutions of systems of ordinary differential equations. These subdominant solutions can be entirely obliterated when integrating with standard numerical techniques if the ever-present error components of the dominant solutions, usually characterized by a strong exponential dependence, increase rapidly in the direction of integration.

The technique presented here, to overcome this difficulty for a fourth-order system, can be made quite efficient by proper use of a functional resembling the linear combination of the Wronskians of two distinct second-order differential equations. This "partial Wronskian" gives a useful piece of global information about the desired subdominant solution which permits the error components of the growing dominant solution to be removed numerically long before these errors have grown unacceptably large.

The technique is presented initially via a specific example taken from the theory of resistive instabilities in an inhomogeneous hydromagnetic fluid. More general problems are discussed briefly to indicate to the reader ways in which the ideas and methods contained in this paper may be modified to treat problems for which a partial-Wronskian functional with the desired properties cannot be found.

I. INTRODUCTION

The techniques of singular perturbation theory [1] and the closely allied phase-integral methods [2] enter into the analysis of many physical problems. Studies of nearly inviscid plane parallel flow governed by the Orr–Sommerfeld Equation ([3], [4]), studies of the Tonks–Dattner Resonances in plasmas with small electron thermal velocities ([5], [6], [7]), and studies of the resistive instabilities of a plasma ([8], [9], [10]) all utilize these techniques to treat differential equations whose highest derivative terms have an asymptotically small coefficient.

The following fourth-order equation helps to illuminate some of the basic elements that these problems have in common and is a good introduction to the

numerical difficulties which often frustrate determination of subdominant solutions. The equations

$$\epsilon^4 \frac{d^4U}{dx^4} + (x^2 + \epsilon^2\lambda) f(x) \frac{d^2U}{dx^2} + g(x)\, U = 0, \tag{1a}$$

$$U(\pm a) = \frac{d^2U}{dx^2}(\pm a) = 0, \tag{1b}$$

specify a well-posed, two-point, boundary-value problem for the eigenvalues $\lambda_i (i = 0, 1,...)$ and the eigenfunctions $U(\lambda_i\,; \epsilon, x)$.[1]

It should be clear that cases where $\epsilon \ll 1$ should be tractable, at least in part, by some simplifying asymptotic approach. In this regard, one treats the four independent solutions of Eq. (1a) as functions of both ϵ and x (and, of course, λ) even though ϵ is usually fixed in any realistic physical application. One then seeks asymptotic expansions of the eigenfunctions to leading order in ϵ which are uniformly valid for all x in the interval $-a$ to $+a$. This approach leads to an "outer-inner-outer-connection" problem in which the interval $-a \leqslant x \leqslant a$ must be split into two entirely different types of regions.

The first type of region is an "outer-limit" region (Oseen-limit [1]) where the derivatives of $U(\lambda; 0, x)$ with respect to x remain of order unity—or are at least asymptotically smaller then $O(1/\epsilon)$. The outer-limit equation for $U(\lambda; 0, x)$,

$$x^2 f(x) \frac{d^2U}{dx^2} + g(x)\, U = 0, \tag{1a'}$$

is singular at $x = 0$ by our previous assumptions. Hence a single outer region cannot span the entire interval $-a \leqslant x \leqslant a$ because derivatives of U become larger than $O(1/\epsilon)$ for $| x |$ sufficiently small.

This singularity signals a region of the second type near $x = 0$, an "inner-limit" region (Stokes-limit[1]), where the fourth-derivative term in Eq. (1a) should not be neglected to leading order. In this region, characterized by $| x | \lesssim \epsilon$, the solutions $U(\lambda; \epsilon, x)$ and also Eq. (1a) are more naturally written as functions of λ, ϵ, and $x^* \equiv x/\epsilon$. The inner-limit equation for $U(\lambda; 0, x^*)$ is then

$$\frac{d^4U}{dx^{*4}} + (x^{*2} + \lambda) f(0) \frac{d^2U}{dx^{*2}} + g(0)\, U = 0. \tag{1a''}$$

In the asymptotic sense, x^* becomes infinite before $x \equiv \epsilon x^*$ becomes finite.

Simplification of the original eigenvalue problem occurs because we can now treat the simplified fourth-order equation (1a″) as an eigen-equation with two boundary conditions specified at $x^* = +\infty$ and two at $x^* = -\infty$. Simplifications

[1] For simplicity we assume that $0 < a \sim O(1)$ and $\epsilon \geqslant 0$. We also assume that $f(x)$ and $g(x)$ are continuous, differentiable, and nonzero for all x in the interval $-a$ to a.

appear in three ways. The parameter ϵ does not appear in Eq. (1a″), the coefficients $f(0)$ and $g(0)$ no longer depend on the independent variable x^* to this order, and the dependence of the eigenfunctions on the boundary condition in the expansion parameter ϵ, and on $f(x)$ and $g(x)$ appear only through the boundary conditions on $U(\lambda; 0, x^*)$ at $x^* = \pm\infty$.[2]

In the inner region there are four independent solutions of Eq. (1a″) which are linear combinations at $x^* = \pm\infty$ of four independent asymptotic expansions. Two of these asymptotic solutions, which we call "dominant" or, more descriptively, "exponential" solutions, have no counterparts in the $\epsilon = 0$ outer-limit equation (unless one counts 0 and ∞ as meaningful solutions). Therefore, uniformity of the final solution of Eq. (1a) over the entire interval $-a \leqslant x \leqslant a$ requires that $U(\lambda; 0, x^*)$ contain none of the exponentially increasing solutions at $\pm\infty$. Any finite amount of the exponentially small solution may be present, however, as this solution matches asymptotically onto zero in the outer regions.

The other two solutions of the inner-limit equation, which we call "subdominant" or, more descriptively, "intermediate" solutions, are neither exponentially small nor exponentially large for large $|x^*|$. These two solutions are the extensions into the inner region of the two independent solutions of the outer-limit equation. The two remaining boundary conditions on the final inner-limit solution, which completes the specification of the simplified eigenvalue problem for λ, specify the ratio of expansion coefficients of the two intermediate-solution asymptotic forms at both $x^* = +\infty$ and $x^* = -\infty$. These two homogeneous boundary conditions ensure a smooth connection between the inner-limit eigenfunction and the two outer-limit solutions which satisfy Eqs. (1b).

These four homogeneous boundary conditions on eigensolutions of the fourth-order equation (1a″) now determine a discrete spectrum of eigenvalues λ. From another point of view, more appropriate to numerical analysis, one specifies three boundary conditions and gives λ, seeking the fourth boundary condition, the ratio of intermediate-solution asymptotic forms, at either $x^* = +\infty$ or $x^* = -\infty$. This numerically computed ratio is then compared with the required ratio to see if the λ value chosen is indeed an eigenvalue.

When Eq. (1a″) can be solved analytically, exact[3] eigenfunctions and eigenvalues

[2] The four boundary conditions on the inner-limit solutions at ∞ are derived from conditions for smooth connection to the outer-limit solutions. This connection, performed at $x^* = \pm\infty$ in the inner region, connects across $x = \pm 0$ for the two outer regions. However, one may also think of this connection as being performed, for a specified nonzero ϵ, at any two points lying within two narrow regions at the outer extremities of the inner region, $1 \ll |x^*| \ll 1/\epsilon$ (or, correspondingly, at the inner edges of the two outer regions).

[3] One should not forget that the eigenvalues of the simplified inner-limit eigenvalue problem are only the leading terms of the exact eigenvalues to Eqs. (1a) and (1b) which have been expanded in asymptotic series in ϵ.

may be found by forming appropriate linear combinations of the four independent solutions. It is not a general property of fourth-order ordinary differential equations with variable coefficients, of which Eq. (1a″) is a particularly simple example, that they be fully soluble in closed form. One may be reduced to solving even this simplified eigenvalue problem numerically.

Unfortunately, numerical integration of Eq. (1a″), or a similar inner-limit equation, for the important intermediate solutions is often hampered by the existence of exponential solutions. This "exponential problem" [9] arises because one of the two exponential solutions increases rapidly in the direction of integration of Eq. (1a″), regardless of which direction is chosen. Any numerical representation of the intermediate solutions will contain relative error components of this rapidly increasing exponential of at least order δ, the relative round-off error of the particular computer in use. This increasing error component can contaminate the intermediate solutions beyond recognition in just a few integration steps when the exponential dependence on x^* is strong.[4]

In some simple cases the exponential problem can be overcome by difference-equation methods or the like which utilize a global property of the solutions, such as symmetry or simultaneous utilization of both boundary conditions. Often, however, a lack of symmetry or of true boundary conditions at both sides of the inner region, or posing the problem as an initial-value problem, makes these rather specialized methods unworkable.

We present a simple step-by-step technique which can be used with quite standard integrating packages for ordinary differential equations. This technique, which we have called the partial-Wronskian technique, utilizes previously determined global information about the exponential solutions, but this information need only be applied locally or quasi-locally insofar as the *intermediate* solutions are concerned. One need not know, for instance, a symmetry relation or the exact asymptotic expansion for $x^* \gg 1$ of the desired intermediate solution whose expansion is posed, as initial data, for $x^* \ll -1$.

In Section II we present this technique through analysis of a specific example. Our technique is made more efficient and elegant in this example by employing a functional resembling a linear combination of the Wronskians of two distinct second-order differential equations. We show how the technique can be applied even when no such partial-Wronskian functional can be derived.

Rather than present a rigorous and detailed treatment of a specific problem, we have only sketched in the mathematical details, which are directly relevant to only a rather restricted class of problems, and have concentrated on explaining the

[4] In this sense we have been calling both exponential solutions "dominant" because their magnitude varies strongly with x^*. Here the magnitude of these two solutions in any region is not as important as the rate of change of this magnitude with x^*.

ideas and methods leading up to the exponential problem and concerned directly in its resolution. We hope that this emphasis on explanation and generality will interest the reader in using and improving our technique for other problems, employing any one of the excellent packages which exist for integrating ordinary differential equations.

II. The Partial-Wronskian Technique—An Example

We present the technique by treating an example taken from a study [10] of the eigenmodes of an inhomogeneous, slightly resistive, magnetohydrodynamic fluid in a finitely sheared magnetic field. A two-point boundary-value problem treated there was reduced to an "outer-inner-outer" connection problem governed by the following fourth-order set of inner-limit equations:

$$\begin{aligned} \frac{d^2\psi(\xi)}{d\xi^2} &= \left[\frac{\gamma}{\lambda^2} + \frac{\xi^2}{\lambda}\right]\psi(\xi) - \xi\phi(\xi), \\ \frac{d^2\phi(\xi)}{d\xi^2} &= \lambda\phi(\xi) - \xi\psi(\xi). \end{aligned} \tag{2}$$

γ is assumed to be finite and real, a given parameter in the problem, and the eigenvalue λ is of order unity and may be complex.

Since these two equations have resisted solution in closed form, we seek uniformly valid approximations to the two intermediate solutions of Eqs. (2) for given values of γ and λ for all $\xi(-\infty < \xi < \infty)$. In the two separate regions $1 \ll |\xi| < \infty$, any solution of Eqs. (2) can be expanded as a linear combination of four independent solutions. The two intermediate solutions have asymptotic expansions whose leading terms are

$$\psi_{\tilde{H}_+}(\xi) \approx \xi^{\beta_+}, \qquad \phi_{\tilde{H}_+}(\xi) \approx \frac{\xi^{1+\beta_+}}{\lambda}, \tag{3a}$$

$$\psi_{\tilde{H}_-}(\xi) \approx \xi^{\beta_-}, \qquad \phi_{\tilde{H}_-}(\xi) \approx \frac{\xi^{1+\beta_-}}{\lambda}, \tag{3b}$$

where[5]

$$\beta_\pm \equiv -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+4\gamma}. \tag{4}$$

[5] In this paper, cases where $\sqrt{1+4\gamma}$ is integral are completely ignored since these rather special values of γ may require a logarithmic solution in place of one of the two asymptotic forms given in Eqs. (3a) or (3b). In addition, we assume that λ does not take on one of the discrete values, $\lambda_l = \sigma_l^2$, where $\mathrm{Re}(\sigma_l) > 0$ and $\sigma_l^3 \equiv (\beta + l + \frac{1}{2}) \pm \sqrt{(\beta + l + \frac{1}{2}) - \gamma}$ for $\beta = \beta_+$ or β_- and for $l = 0, 2, 4, \ldots$. In [10] it is shown that certain of our assumptions concerning the nature of the two intermediate solutions are wrong for these specific values of λ. For these particular values, in fact, the outer-inner-outer connection has been performed analytically.

The two exponential solutions have asymptotic expansions starting with the terms

$$\psi_{\tilde{x}_+}(\xi) \approx \xi^{\mu_+} \exp[\xi^2/(4\lambda)^{1/2}], \qquad \phi_{\tilde{x}_+}(\xi) \approx -\lambda\xi^{-1+\mu_+} \exp[\xi^2/(4\lambda)^{1/2}], \tag{3c}$$

$$\psi_{\tilde{x}_-}(\xi) \approx \xi^{\mu_-} \exp[-\xi^2/(4\lambda)^{1/2}], \qquad \phi_{\tilde{x}_-}(\xi) \approx -\lambda\xi^{-1+\mu_-} \exp[-\xi^2/(4\lambda)^{1/2}] \tag{3d}$$

In Eqs. (3c) and (3d) we use the definition

$$\mu_\pm \equiv \tfrac{1}{2}[\pm\gamma\lambda^{-3/2} - 1 \pm \lambda^{3/2}]. \tag{5}$$

The four asymptotic expansions for ξ large and negative are found from the positive-ξ solutions, Eqs. (3), by using the intrinsic symmetry of Eqs. (2).

We stress that the solutions (3) are only asymptotic expansions of exact solutions.[6] Each of the four exact solutions, in general, matches onto a linear combination of the four expansions (3) for $\xi \gg 1$ and onto a different linear combination for $\xi \ll 1$. However, only the expansion coefficient of the truly dominant asymptotic expansion can be known in these linear combinations because the error in the exponentially large asymptotic form is as large as or larger than the other three solutions in principle. This asymptotic dominance, as we will show, is not only the source of the exponential problem but also the tool by which it is resolved.

The growth factor of the exponential expansions (3c) and (3d),

$$G(\xi_2, \xi_1) \equiv \exp\left|\frac{\xi_2^2 - \xi_1^2}{2\sqrt{\lambda}}\right|, \tag{6}$$

measures the approximate change in magnitude of the exponential solutions between the points ξ_1 and ξ_2. This growth factor is particularly vicious when $|\lambda| \ll 1$. If, for example, we integrate from $\xi_1 = 12$ to $\xi_2 = 13$ with $\lambda = 0.01$, the two exponential solutions change magnitude by a factor of order exp(125) while

[6] Let $\mathbf{A}(\xi)$ denote a vector whose four components, $\psi_A(\xi)$, $\psi'_A(\xi)$, $\phi_A(\xi)$, $\phi'_A(\xi)$, are the values of one of the independent solutions of Eqs. (2) for a given γ and λ. $\mathbf{A}(\xi_0)$ contains sufficient information to determine $\mathbf{A}(\xi)$ for all ξ in principle by integrating Eqs. (2) from ξ_0 to ξ. If we have at our disposal four independent solutions of (2) for given λ and γ—call them $\mathbf{A}_1(\xi)$, $\mathbf{A}_2(\xi)$, $\mathbf{A}_3(\xi)$, $\mathbf{A}_4(\xi)$—any solution of Eqs. (2) can be written as

$$\mathbf{a}(\xi) = \sum_{i=1}^{4} c_i \mathbf{A}_i(\xi),$$

and the expansion coefficients c_i can be determined when $\mathbf{a}(\xi_0)$ is given.

In addition to exact solutions, we wish to discuss both asymptotic expansions of exact solutions and numerical approximations to exact solutions. The former we denote by a tilde, the latter by a bar. Thus $\mathbf{a}(\xi) \approx \tilde{\mathbf{a}}(\xi)$ in some appropriate asymptotic sense and $\mathbf{a}(\xi) \approx \bar{\mathbf{a}}(\xi)$ in the numerical sense.

the intermediate solutions remain of order unity. A computer would need more than fifty decimal digits of precision to make the relative round-off error δ of order $\exp(-125)$ in order that initial error components of the spurious growing exponential solution be kept small.

The technique developed to avoid the exponential problem involves swimming with the tide rather than against it. When seeking the fastest growing solution, direct uncorrected numerical integration is adequate since all spurious components of the other three solutions, which arise from numerical error of one sort or another, are suppressed relative to the fastest growing solution during an integration step. Once we have found the two rapidly varying exponential solutions quite accurately, they can be used in a simple step-by-step procedure to "orthogonalize" the numerically determined intermediate solutions against the exponential problem. Our technique for this specific example employs a functional, which we call a partial Wronskian, derived directly from Eqs. (2).

$$\begin{aligned} W[\mathbf{a}(\xi), \mathbf{b}(\xi)] &\equiv W_{ab}(\xi) \\ &\equiv [\psi_a(\xi)\,\psi_b'(\xi) - \psi_b(\xi)\,\psi_a'(\xi) + \phi_a(\xi)\,\phi_b'(\xi) - \phi_b(\xi)\,\phi_a'(\xi)] \\ &= \text{constant}, \end{aligned} \tag{7}$$

where $\mathbf{a}(\xi)$ and $\mathbf{b}(\xi)$ are any two solutions of Eqs. (2) for given γ and λ. Clearly $W_{aa} = 0$ and $W_{ab} = -W_{ba}$.

Let $\mathbf{S}(\xi)$ and $\mathbf{L}(\xi)$ be any two solutions of Eqs. (2) which are normalized so that

$$W_{SL}(\xi) = 1 \qquad (-\infty < \xi < \infty). \tag{7a}$$

The other two solutions, denoted by $\mathbf{I}_1(\xi)$ and $\mathbf{I}_2(\xi)$, can then be chosen so that[7]

$$\begin{aligned} &W_{I_1S} = W_{I_2S} = W_{I_1L} = W_{I_2L} = 0, \\ &W_{I_1I_2} = 1 \qquad (\text{for } -\infty < \xi < \infty). \end{aligned} \tag{7b}$$

Now let $\mathbf{S}(\xi)$ be the solution of Eqs. (2) which is exponentially small, varying asymptotically as Eq. (3d), for $\xi \gg 1$. The numerical approximation $\bar{\mathbf{S}}(\xi)$ can be

[7] It is clear that an $\mathbf{I}_1(\xi)$ and an $\mathbf{I}_2(\xi)$ satisfying (7b) always exist, but the partial-Wronskian functional itself, Eq. (7), is only particularly useful when the asymptotic expansions of Eqs. (3) satisfy Eqs. (7b) term by term with $\mathbf{S}$ and $\mathbf{L}$ replaced by $\tilde{\mathbf{x}}_+$ and $\tilde{\mathbf{x}}_-$ and with $\mathbf{I}_1$ and $\mathbf{I}_2$ replaced by $\tilde{\mathbf{H}}_+$ and $\tilde{\mathbf{H}}_-$. Although the asymptotic expansions do satisfy (7b) for our present example, the usefulness of our technique is not restricted to such cases. A generalized problem, where a simple functional such as (7) with asymptotic expansions satisfying (7b) need not be derived, is treated briefly in the next section.

found accurately by integrating Eqs. (2) from large positive ξ_0 toward $-\infty$ for this is the direction of growth for $\bar{\mathbf{S}}(\xi)$.[8]

Similarly, $\bar{\mathbf{L}}(\xi)$, our approximation to $\mathbf{L}(\xi)$, is made exponentially small for $\xi \leqslant -\xi_0$ and found for $\xi > -\xi_0$ by integration of Eqs. (2) toward $+\infty$. For both $\bar{\mathbf{S}}$ and $\bar{\mathbf{L}}$, the numerical starting values, $\bar{\mathbf{S}}(\xi_0)$ and $\bar{\mathbf{L}}(\xi_0)$, are taken directly from Eq. (3d). After integration, the two functions thus determined are normalized to ensure that $W_{\bar{S}\bar{L}}(\xi) = 1$.

To tabulate $\bar{\mathbf{I}}_1(\xi)$ and $\bar{\mathbf{I}}_2(\xi)$, adequate numerical approximations to the two intermediate solutions, the functions $\bar{\mathbf{S}}(\xi)$ and $\bar{\mathbf{L}}(\xi)$ and the functional W are used in the following step-by-step procedure.

Suppose we know the four values $\bar{\mathbf{I}}(\xi_1)$.[9]

(1) Choose an integration step length $\delta\xi > 0$ such that $G(\xi_1 + \delta\xi, \xi_1) \ll 1/\delta$.

(2) Choose a positive integer N such that $G(\xi_1 + N\delta\xi, \xi_1) \gg 1/\delta$.

(3) Integrate Eqs. (2) from ξ_1 to $\xi_2 \equiv \xi_1 + \delta\xi$ using $\bar{\mathbf{I}}(\xi_1)$ as initial values. Employ any ordinary differential-equation integration package which satisfies appropriate error criteria and then retain these new values as a trial intermediate solution $\bar{\mathbf{I}}^*(\xi_2)$.

(4) Integrate forward another $N - 1$ steps from ξ_2 to $\xi_3 \equiv \xi_1 + N\delta\xi$ using $\bar{\mathbf{I}}^*(\xi_2)$ as initial values.

(5) The *accurate* approximation to $\mathbf{I}(\xi)$ is then determined *at* ξ_2 by the relation

$$\bar{\mathbf{I}}(\xi_2) = \bar{\mathbf{I}}^*(\xi_2) - W_{\bar{S}\bar{I}^*}(\xi_3)\,\bar{\mathbf{L}}(\xi_2). \tag{8}$$

(6) The entire process, starting at step (1), is repeated over and over until $\xi = +\xi_0$ is reached. The solution $\bar{\mathbf{I}}(\xi_0)$ can then be treated entirely in terms of the appropriate linear combination of asymptotic expansions for $\xi > \xi_0$.

This technique of corrected integration was actually used to solve the eigenvalue problem given by Eqs. (2) with appropriate boundary conditions. Figure 1 shows a typical computation with and without the correction steps (4), (5), and (6). With $\lambda = 3.0$, the exponents of the exponential solutions are not particularly large—see Eq. (6)—but the exponential problem is so severe, even at its mildest, that the step-by-step corrections are absolutely imperative. (In [10], meaningful computa-

[8] We pick $\xi_0 \gg 1$ to be so large that all four asymptotic expansions (3) for $|\xi| \geqslant \xi_0$ are more accurate than the relative numerical error δ from round-off and truncation. Then we need consider only the region $-\xi_0 \leqslant \xi \leqslant \xi_0$ numerically because the appropriate linear combinations of asymptotic expansions are entirely adequate to describe the numerical approximations for $|\xi| \geqslant \xi_0$.

[9] The procedure is initialized by setting $\bar{\mathbf{I}}_1(\xi_0) = \tilde{\mathbf{H}}_+(\xi_0)$ and $\bar{\mathbf{I}}_2(\xi_0) = \tilde{\mathbf{H}}_-(\xi_0)$ from Eqs. (3a) and (3b). We use $\bar{\mathbf{I}}$ to denote either of these two approximate solutions and note that $\mathbf{L}(\xi)$ is the undesired exponential solution, because we intend to integrate toward $\xi = +\infty$.

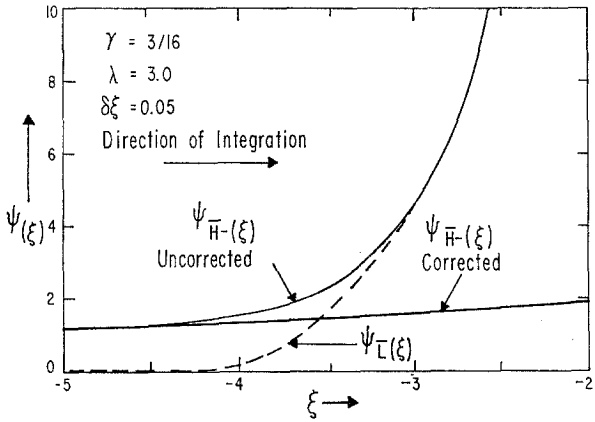

FIG. 1. Corrected and uncorrected integrations testing the partial-Wronskian technique. Without correction the numerical solution, $\psi_{\bar{H}\_}(\xi)$, departs increasingly from the desired intermediate solution and rapidly becomes dominated by the undesired exponential solution, $\psi_{\bar{L}}(\xi)$.

tions were carried out for λ as small as 0.03.) Without correction, the numerical solution deviates rapidly from the intermediate solution and quickly approaches the exponential solution, $\bar{\mathbf{L}}(\xi)$. With correction, however, certain global checks indicate that the relative overall errors were kept as small as 10^{-3} to 10^{-4} for $-\infty < \xi < \infty$.

III. The Partial-Wronskian Technique—An Explanation and Generalization

An understanding of why the technique works is prerequisite to applying the technique to other problems for which a useful partial-Wronskian functional is not forthcoming. In such problems it will be necessary to derive a modified correction formula to replace Eq. (8), but this should always be possible whenever uncorrected numerical integrations suffer a severe exponential problem.

Consider the steps which lead up to and utilize the correction formula (8). At $\xi = \xi_2$ the numerical trial solution $\bar{\mathbf{I}}^*(\xi_2)$ is a linear combination of all four exact solutions. We write this combination as

$$\bar{\mathbf{I}}^*(\xi_2) = \mathbf{I}(\xi_2) + \delta^*\mathbf{L}(\xi_2) + \cdots, \tag{9}$$

where the two exact intermediate-solution components are combined in $\mathbf{I}(\xi_2)$ and where the $\mathbf{S}(\xi_2)$ component is neglected because $\mathbf{S}$ is decreasing in the direction of integration. Because $\mathbf{L}(\xi)$ has increased rapidly during the integration of step (3)

above—from $\xi = \xi_1$ to $\xi = \xi_2$—the major numerical error, $\delta^*\mathbf{L}(\xi_2)$, is no longer of order δ, the relative computer round-off error, but has become of order $\delta G(\xi_2, \xi_1)$. This increased relative error is now much larger than order δ (but still much less than unity if $\delta\xi \equiv \xi_2 - \xi_1$ is properly chosen). Further, for continued integration of $\bar{\mathbf{I}}^*$ beyond ξ_2, as in step (4) above, δ^* is essentially constant, varying by no more than relative terms of order $1/G(\xi_1, \xi)$, since the spurious exponential component has already become dominant.

The entire point of our technique, therefore, is to determine δ^* in Eq. (9) and then to reduce the error in $\bar{\mathbf{I}}^*$ back to relative order δ. The method of determining δ^* is relatively unimportant.

To find δ^*, step (4) is carried out. By the time $\xi = \xi_3$ is reached, $G(\xi_3, \xi_1) \gg 1/\delta$, and thus $\bar{\mathbf{I}}^*(\xi_3)$ is dominated by the exponential solution $\bar{\mathbf{L}}(\xi_3)$. We have

$$\bar{\mathbf{I}}^*(\xi_3) \approx \delta^*\bar{\mathbf{L}}(\xi_3). \tag{10a}$$

In the specific example of the last section we were fortunate to have the functional W available so that δ^* could be extracted explicitly by forming the partial Wronskian,

$$W_{S\bar{I}^*}(\xi_3) \approx \delta^*\bar{\mathbf{L}}W_{S\bar{L}}(\xi_3) \approx \delta^*. \tag{10b}$$

Thus it should now be obvious that Eq. (8) is nothing more than the subtraction from the trial intermediate solution of the exponentially increasing error term *before* it has grown unacceptably large. We stress that W plays a useful but not necessary role in the example chosen. The integer N of steps (2) and (4) can be unity in any problem where a functional with the properties of W can be constructed. Thus W permitted us to determine δ^* and exorcise the error terms long before these became large, a saving of many unnecessary integration steps.

In the absence of a functional with the properties of W, N must be larger than unity, and Eq. (10a) must be used to find δ^*. Thus

$$\delta^* \approx \psi_{\bar{I}^*}(\xi_3)/\psi_{\bar{L}}(\xi_3) \approx \phi_{\bar{I}^*}(\xi_3)/\phi_{\bar{L}}(\xi_3) \approx \cdots. \tag{11}$$

The value of δ^* from Eq. (11), which can be used in the form

$$\bar{\mathbf{I}}(\xi_2) = \bar{\mathbf{I}}^*(\xi_2) - \frac{\psi_{\bar{I}^*}(\xi_3)}{\psi_{\bar{L}}(\xi_3)}\bar{\mathbf{L}}(\xi_2) \tag{12}$$

to replace Eq. (8) entirely, is just as accurate as the value from Eq. (10b), although it does take longer to evaluate.

It is remarkable, perhaps, that δ^* can be accurately determined precisely because there is an exponential problem rather than in spite of it. The source of the difficulties also provides the means by which the difficulty may be overcome.

To generalize Eq. (2) to a fourth-order system, where a functional with all the properties of W is not available, consider, for example, the equations

$$\begin{aligned} \frac{d}{d\xi}\left[a_1(\xi)\frac{d\psi}{d\xi}\right] &= f_1(\xi)\,\psi + g_1(\xi)\,\phi, \\ \frac{d}{d\xi}\left[a_2(\xi)\frac{d\phi}{d\xi}\right] &= g_2(\xi)\,\psi + f_2(\xi)\,\phi, \end{aligned} \tag{13}$$

in which we assume that all of the coefficient functions in Eqs. (13) are of such a nature that an exponential problem exists.

We again expect no difficulty in determining two solutions—$\bar{\mathbf{S}}(\xi)$ and $\bar{\mathbf{L}}(\xi)$—which are exponentially small for $\xi \gg 1$ and $\xi \ll 1$, respectively. These can be determined accurately, in fact, even when the four asymptotic expansions for $|\xi| \geqslant \xi_0 \gg 1$ are entirely unknown.

Then, using Eq. (12) in place of (8), the two intermediate solutions can be determined in the region $|\xi| \leqslant \xi_0$ by the step-by-step technique outlined above. In cases where the intermediate-solution asymptotic expansions are unknown for $|\xi| > \xi_0$, starting values for $\bar{\mathbf{I}}(-\xi_0)$ can easily be determined numerically. Let us set

$$\bar{\mathbf{I}}^*(-\xi_0) = (1, 0, 0, 0)$$

provisionally [$(\psi, \psi', \phi, \phi')$ as indicated earlier]. This defines a solution function containing both $\bar{\mathbf{S}}(\xi)$ and $\bar{\mathbf{L}}(\xi)$, as well as some linear combination of the two intermediate solutions. It can therefore be written as

$$\bar{\mathbf{I}}^*(-\xi_0) = \bar{\mathbf{I}}(-\xi_0) + \delta_S\bar{\mathbf{S}}(-\xi_0) + \delta_L\bar{\mathbf{L}}(-\xi_0). \tag{14}$$

As in our earlier discussion, δ_S and δ_L can be determined by integrating $\bar{\mathbf{I}}^*(-\xi_0)$, first to the left and then to the right of $-\xi_0$. Using the known functions $\bar{\mathbf{S}}$ and $\bar{\mathbf{L}}$, the component of $\bar{\mathbf{I}}^*(-\xi_0)$, which is a pure intermediate solution, can be isolated by extracting both of the exponential terms. A second independent intermediate-solution starting vector can be found, as above, by provisionally setting[10]

$$\bar{\mathbf{I}}^*(-\xi_0) = (0, 1, 0, 0).$$

To demonstrate this generized technique for extracting subdominant solutions we have treated a simple example which is both analytically soluable and subject to the exponential problem. The two equations

$$\begin{aligned} \frac{d^2\psi(\xi)}{d\xi^2} &= b^2\phi(\xi), \\ \frac{d^2\phi(\xi)}{d\xi^2} &= b^2\psi(\xi), \end{aligned} \tag{15}$$

[10] In both of these initial value determinations we ignore the possibility that (1, 0, 0, 0) or (0, 1, 0, 0) is proportional to one of the two exponential solutions.

have four independent solutions,

$$\psi_L(\xi) = e^{b\xi}, \qquad \psi_s(\xi) = e^{-b\xi},$$
$$\psi_{H+}(\xi) = \cos b\xi, \qquad \psi_{H-}(\xi) = \sin b\xi. \tag{16}$$

We approached Eqs. (15) numerically, proposing to integrate the solution $\psi_{H-}(\xi)$ from $\xi = -4$ to $\xi = +4$. We set $b = +10$ so that the large solution, growing by a factor 2×10^4 for each unit of ξ integrated in the positive-ξ direction, clearly introduces an exponential problem.

This example is particularly illuminating because we chose to use the generalized technique even though a partial Wronskian exists for this problem. Thus we employed overintegration, that is, integration far enough past the desired connection point to permit the component of the large solution, $\exp(10\xi)$, to grow much larger than the desired subdominant solution, $\sin(10X)$.

The integration was done using an extrapolation-integrator developed by one of the authors and N. K. Winsor of the Princeton Plasma Physics Laboratory. A relative error tolerance of 1 in 10^5 was required of the integrator. The differential equations (15) were integrated first from $\xi = -4.0$ to $\xi = 6.0$ using initial conditions appropriate to the large solution, $\psi_L(\xi) = \exp(10X)$. The integration was then repeated for the desired subdominant solutions. The trial solution after the 30th step, each of length 0.1, was then divided by the numerically determined large solution at that point to give δ^* of Eq. (11). The first 10 steps were then corrected, and the corrected values of $\psi_{H-}(\xi)$ at the tenth step were used to restart the procedure.

Figure 2 shows the results. The dashed lines show segments of the overintegrations that were performed, the trial integrations of Step (4). The

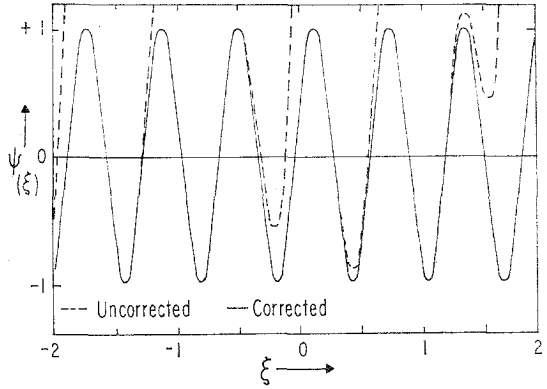

FIG. 2. Corrected and uncorrected integrations demonstrating the generalized partial Wronskian technique. Without correction the numerical solution (dashed lines) departs increasingly from the exact solution, $\sin 10X$.

function grows rapidly as the error component, $\exp(10\xi)$, takes hold. The solid line is the corrected numerical approximation to $\psi_{H-}(\xi) = \sin(10X)$. At $\xi = +4$, not shown in the figure, there were still between four and five decimal digits of precision in the corrected numerical approximation $\psi_{H-}(\xi)$.

Although our partial-Wronskian technique in its most elegant and efficient form might seem to apply to only a limited class of eigenvalue equations for which a functional analogous to W with the required properties can be found, the baser form, which requires extended sub-integrations to determine δ^* for each integration step, seems to be very general indeed. We have only scratched the surface here, leaving undiscussed many relevant questions—such as extension to higher-order systems, analysis of error accumulation, and optimum step lengths. Rather than in-depth analysis, our goal has been to present the basic methods and ideas of the technique in an understandable way. We hope that the reader will find the opportunity and inclination to attack other problems using these ideas.

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